Combinatorial expression for universal Vassiliev link invariant(GEOMETRIC ASPECTS OF INFINITE ANALYSIS)

AUTHOR(S):

Piunikhin, Sergey

数理解析研究所講究録
第 883 巻 1994 年 111-133

HARVARD UNIVERSITY
DEPARTMENT OF MATHEMATICS

Main Office (617) 495-2171
Undergraduate Studies Coordinator (617) 495-9116
Graduate Studies Coordinator (617) 495-2170
Telex 62006659
Fax (617) 495-5132

Science Center 325
One Oxford Street
Cambridge, Massachusetts 02138
Internet: math.harvard.edu
Bitnet: huma1.bitnet

Combinatorial expression for universal Vassiliev link invariant

Sergey Piunikhin

March 22 , 1993

Harvard University
Department of Mathematics
Cambridge, MA 02138 USA
e-mail serguei@math.harvard.edu

Abstract

A general model similiar to R-matrix-type models for link invariants is constructed. It contains all R-matrix invariants and is a generating function for "universal" Vassiliev link invariants. This expression is simpler than Kontsevich's expression for the same quantity, because it is defined combinatorially and does not contain any integrals, except for an expression for "the universal Drinfeld's associator".

1. INTRODUCTION

Vassiliev knot invariants were invented in attempts to construct some natural basis for the space of all knot invariants (this space can be described as the cohomology space H^0(Imbeddings:$S^1 \to R^3$)). For this purpose Vassiliev used stratification of the discriminant set of nonimbeddings: $S^1 \to R^3$ (by the number of double transversal crossings) and some finite-dimensional approximations of the space of all knots. (We recommend the reader [Va1], [Va2] and expecially [BN1] for a very detailed introduction to the theory of Vassiliev invariants).

Although the question whether Vassiliev knot invariants can distinguish any two knots is still open, this language seeems to be the most appropriate in studying classical knot and link invariants.

All known classical knot and link invariants : Alexander polynomial, Jones polynomial, Kauffman polynomial, HOMFLY polynomial and all their generalizations, as well as Milnor μ-invariants (see [Ro], [Co], [Jo1], [Ka1], [Ka2], [HOMFLY],[Tu1],[Tu2],[Re1],[RT],[Mi1], [Mi2] for a precise definitions), can be incorporated into this scheme (see [BL],[Li1],[Li2],[BN5]).

The space of Vassiliev knot invariants of fixed order n (divided by the space of invariants of order $n-1$) has a purely combinatorial description. It is isomorphic to a certain linear subspace in the space of functions on the set of "Vassiliev $[n]$-diagrams" (or combinatorial types of n pairs of points on S^1). The linear relations, defining this subspace in the space of all functions on the set of "Vassiliev $[n]$-diagrams" were first written explicitly by Birman and Lin [BL]. The fact, that the set of relations written in [BL] is complete and there are no extra relations, was proved by Kontsevich [Ko1] .

To prove the isomorphism between the space of Vassiliev knot invariants of order n (divided by the space of invariants of order $n-1$) and the linear space F_n^*, defined purely combinatorially, Kontsevich used an explicit integral presentation of "the universal Vassiliev invariant of order n".This "universal invariant" I_n takes values in the linear space F_n , dual to F_n^* .

The space F_n has another very nice description in terms of Feynman diagrams of perturbative Chern-Simons theory [BN1] . The graded linear space $F=\oplus_n F_n$ admits a Hopf algebra structure [Ko1], [BN3] (Kontsevich Hopf algebra). Th space of primitive elements in this Hopf algebra is generated by connected Feynman diagrams [Pi2].

The generating function $I=\sum_{n=0}^{\infty} h^n I_n$ of "the universal Vassiliev invariants of order n"gives us "the universal Vassiliev invariant" I taking its values in Kontsevich Hopf algebra F. Here h is formal parameter, $I_n(K)$ is certain n-fold integral over the knot K (Kontsevich integral [Ko1],[Ar2]). At the moment nobody is able to calculate explicitly $I(K)$ for any non-trivial knot K.

The aim of the present paper is to give a simpler expression for this quantity, which can be calculated explicitly to all orders in h if one can calculate "the universal Drinfeld's associator" [Dr1] . This expression models state sum expression for knot polynomials $P_{g,V}(q^{\pm 1})$ (here $q=e^h$) constructed from a simple Lie algebra g and its irreducible representation V (see [Re1], [Tu1], [Jo2] for an explicit form of this state sum expression) .

Connection between $P_{g,V}(q^{\pm 1})$ and Vassiliev knot invariants was found in the most general form by Lin [Li1] : If $P_{g,V}$(h)= $\sum_{n=0}^{\infty} P_{g,V,n} h^n$ than $P_{g,V,n}$ is Vassiliev invariant of order n . Explicit state sum expression for $P_{g,V,n} \in F_n$ was deduced in [Pi1].

The question is, whether it is possible to forget about the Lie algebra g and the representation V and to write the "universal" state sum expression $P=\sum_{n=0}^{\infty} h^n P_n$ with values in Kontsevich Hopf algebra F .

There are two ways to do this. The first one (using complicated integrals) was found by Kontsevich [Ko1]. The second way (combinatorial) is presented here.

The paper is organized as follows:

In section 2 the basic facts about Vassiliev link invariants are presented.

In section 3 Drinfeld's construction of "the universal prounipotent" braid group representation is presented.

In section 4 F-valued "Markov trace" in this representation is constructed and the fact that it is a generating function for "universal Vassiliev link invariants" is proved. Multiplicative property of this "universal invariant" with respect to connected sums is proved . Generalization for string link invariants is also given.

In section 5 some open problems are discusseed.

2. PRELIMINARIES

Definition.

We shall call a trivalent graph consisting of several directed circles (called Wilson loops) and several dashed lines (called propagators) a **CS-diagram**. The propagators and Wilson loops are allowed to meet in two types of vertices: one type (called R^2g-vertices) in which a propagator ends on one of the Wilson loops; and another type (called g^3 -vertices) connecting three propagators.

We assume, that one of two possible cyclic orders of propagators meeting in any g^3-vertex is specified.

Each CS-diagram can be uniquely presented by its plane projection (see fig.1. as an example)

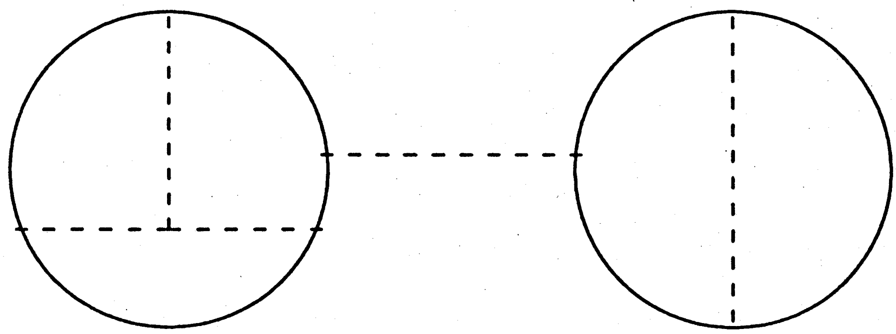


fig.1

Here, as usual, we assume, that the counterclockwise cyclic order in each g^3 -vertex is fixed. For instance, the cyclic orders of propagators on graphs in fig.2a and fig.2b are different.

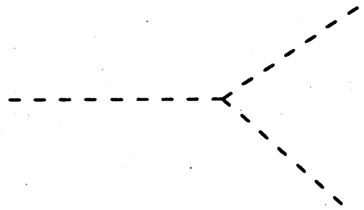


fig. 2a

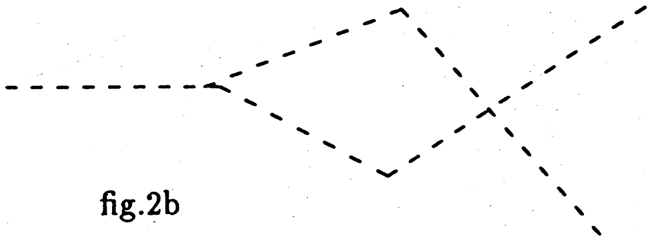


fig.2b

Let K be some ring $Z \subset K \subset C$.

Definition.

A function $C : (CS - diagrams) \rightarrow K$ is called a **weight system** if

$$C(S) = C(T) - C(U) \tag{2.1}$$

where S, T, U are CS-diagrams, identical everywhere except in some small ball, where they look as in fig.3

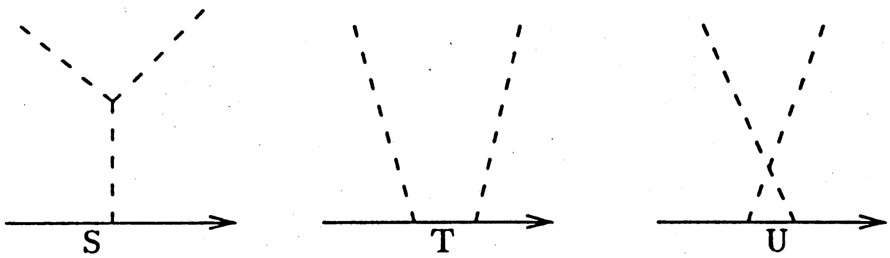


fig. 3

Definition.

Following Vassiliev [Va1], [Va2] and Birman-Lin [BL] we shall call a CS-diagram with $2n$ R^2g-vertices and without g^3 -vertices a **Vassiliev [n]-diagram**, and a CS-diagram with $2n-2$ R^2g-vertices and with one g^3-vertex a **Vassiliev $< n >$-diagram** .

Let D be Vassiliev $< n >$-diagram. Let z_1 , z_2 and z_3 be three R^2g-vertices connected by propagators with (the unique) g^3-vertex in D. Let us define Vassiliev $[n]$-diagrams D_{1+} , D_{1-} as Vassiliev $[n]$-diagrams, obtained from D by the local procedure shown in fig.4

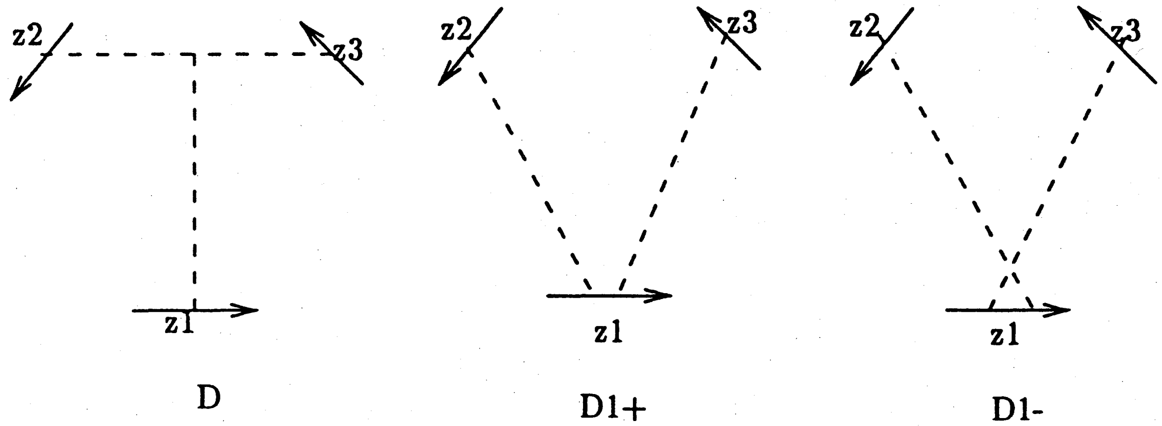


fig.4

(The Vassiliev $[n]$-diagrams D_{2+} , D_{2-} , D_{3+} , D_{3-} can be defined in the same way by changing z_1 to z_2 and to z_3 respectively).

Definition. *[BL], [Va1], [Va2], [Ko1].*

Let $W_n^s(s \in N)$ be a free K-module, generated by the set of s-Wilson-loop Vassiliev $[n]$-diagrams; let F^s be the quotient of W_n^s by the ideal, generated by relations

$$D_{1+} - D_{1-} = D_{2+} - D_{2-} \tag{2.2}$$

(D runs over Vassiliev $< n >$-diagrams). Let us denote $F_0^s = K$;

$F^s = \oplus_n F_n^s$, and let us identify $1 \in K = F_0^s$ with (the unique) s-Wilson-loop Vassiliev [0]-diagram.

Theorem 2.1. *[Ko1], [Ar1], [BN1].*

K-module F_n^s is isomorphic to the quotient of the free module D_n^s , generated by s-Wilson-loop CS-diagrams with Euler characteristics $1 - n - s$ by the ideal, generated by relations $(2.3) - (2.5)$

$$S = T - U, \tag{2.3}$$

where S, T and U are CS-diagrams, identical everywhere except in some small ball, where they look as in fig.3 .

$$I = H - X, \tag{2.4}$$

where I, H and X are CS-diagrams, identical everywhere except some small ball, where they look as in fig.5

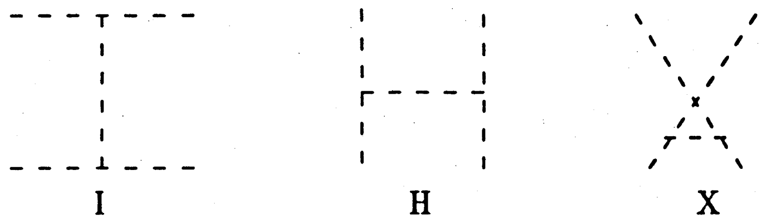


fig.5

$$Y + Z = 0, \tag{2.5}$$

where Y and Z are CS-diagrams, identical everywhere except some small ball, where they look as in fig.2a and 2b respectively.

In fact F^1 can be equipped with a structure of a graded Hopf algebra [Kol] and we shall call it the Kontsevich Hopf algebra. When it will not lead to confusion, we'll omit the superscript 1 and write $F = F^1$.

The Kontsevich Hopf algebra F acts on F^s (taking connected sum along Wilson loop) in s different mutually commuting ways [BN5] , thus we have a graded action of $F^{\otimes s}$ on F^s.

Let $A = \oplus_n A_n$ be the quotient of Kontsevich algebra by the ideal generated by F_1 . (The K-module F_1 has rank one and is generated by a single Vassiliev [1]-diagram. Let us denote this diagram by $t \in F_1$). Since the element t is primitive, A is also a Hopf algebra.

It is well-known [Kol] that the space A_n^* dual to A_n is canonically isomorphic to the space V_n of Vassiliev knot invariants of order n factored by the space V_{n-1}. The map $V_n/V_{n-1} \to A_n^*$ is the evaluation of a knot invariant on singular embeddings with n double points [Va1], [Va2], [BL] which gives a linear function $V_n/V_{n-1} \otimes A_n \to C$.

The inverse map $I_n : A_n^* \to V_n \to V_n/V_{n-1}$ was first constructed in [Kol] and is called "Kontsevich integral". The aim of this paper is to construct (formally another) inverse map $P_n : A_n^* \to V_n$ which has a simple combinatorial description.

Definition.

Let X^m $(m \in N)$ be the graded completion of Lie algebra $\oplus_n X_n^m$, with generators t^{ij} (i<j) of degree 1 and with relations

$$[t^{ij}; t^{kl}] = 0 \qquad (i \neq j \neq k \neq l) \tag{2.6A}$$

$$[t^{ij}; t^{ik} + t^{jk}] = 0 \tag{2.6B}$$

The universal enveloping algebra UX^m of this Lie algebra is prounipotent completion of the group algrbra of the pure braid group (see [K2] and references therein). Kohno [K1] used this algebra in order to write the most general form of Knizhnik-Zamolodchikov equation [KZ]

$$\frac{d\Psi}{dz_i} = \hbar \sum_{j\neq i} \frac{t^{ij}}{z_i - z_j} \Psi, \tag{2.7}$$

where ψ is a UX^m-valued meromorphic function on $(C^m \backslash diagonals)$, $\hbar = \frac{h}{2\pi i}$. Relations (2.6) are imposed in order to preserve the zero-curvature condition

$$[\frac{d}{dz_i} - \hbar \sum_{j\neq i} \frac{t^{ij}}{z_i - z_j}; \frac{d}{dz_k} - \hbar \sum_{l\neq k} \frac{t^{kl}}{z_k - z_l}] = 0 \tag{2.8}$$

which allows us to construct monodromy representation of pure braid group in the group $exp(X^m)\imath UX^m$. This representation is nonlocal and its matrix elements are certain hypergeometric-type integrals (see [Ao] and [K2] for more detailed exposition. In our approach we don't use this complicated technics).

Algebra UX^m can be imbedded in the algebra A^m_{kz} of Feynman diagrams (see [BN1],[BN5]) of the form depicted on fig 6 .

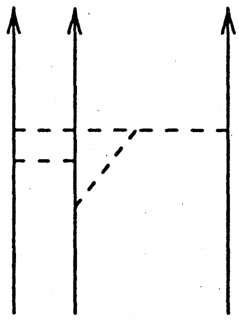


fig.6

These diagrams are defined in the same way as usual CS-diagrams, but they have m upward pointed Wilson lines instead of one Wilson loop. Here t^{ij} is presented by the diagram on fig.7.

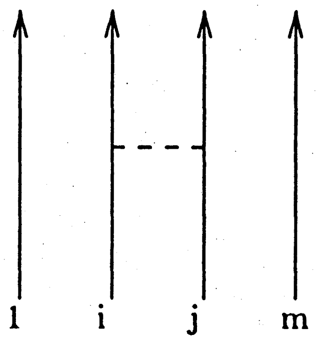


fig.7

The diagram with $2n$ vertices (this number is always even) is said to be of degree n. The multiplication in the algebra A^m_{kz} of diagrams is just putting one diagram over another. It is easy to see that the grading and multiplication in A^m_{kz} defined above are compatible with those in UX^m .

3. Explanations of Drinfeld's construction

Let K be some field. Let $\phi(A, B)$ be some formal power series in two noncommuting variables A and B with the coefficients in K and let

$$\Phi = \phi(\hbar t^{12}, \hbar t^{23}) \in UX^3 \tag{3.1}$$

Definition.

The formal noncommutative power series $\phi(A, B)$ will be called **associator** if $log(\phi(A, B))$ belongs to the graded completion of the free Lie algebra with two generators A and B and if the equations (3.2) – (3.5) hold:

$$\phi(\hbar t^{12}, \hbar(t^{23} + t^{24}))\phi(\hbar(t^{12} + t^{13}), \hbar t^{34}) =$$
$$= \phi(\hbar t^{23}, \hbar t^{34})\phi(\hbar(t^{12} + t^{13}), \hbar(t^{24} + t^{34}))\phi(\hbar t^{12}, \hbar t^{23}) \in exp(X^4) \tag{3.2}$$

$$e^{\frac{\hbar t^{13} + \hbar t^{23}}{2}} = \Phi^{312} e^{\frac{\hbar t^{13}}{2}} (\Phi^{132})^{-1} e^{\frac{\hbar t^{23}}{2}} \Phi \in exp(X^3) \tag{3.3}$$

$$e^{\frac{\hbar t^{13} + \hbar t^{12}}{2}} = (\Phi^{231})^{-1} e^{\frac{\hbar t^{13}}{2}} \Phi^{213} e^{\frac{\hbar t^{12}}{2}} (\Phi)^{-1} \in exp(X^3) \tag{3.4}$$

$$\Phi^{321} = \Phi^{-1} \in exp(X^3) \tag{3.5}$$

Here Φ^{ijk} (ijk is a permutation of 123) is the image of $\Phi \in UX^3$ under automorphism

$$s_{ijk} : UX^3 \to UX^3$$

which maps t^{12} to t^{ij} ; t^{13} to t^{ik} and t^{ij} to t^{jk} .

Theorem 3.1 (Drinfeld).

The "associator" exists for any field K such that $Q \subset K \subset C$.

We'll give here an explicit construction of associator for $K = C$ due to Drinfeld. This construction will not be used later. We'll need for our purposes only formal properties (3.2) – (3.5) of "associator" $\phi(A, B)$ but not an explicit form of this "associator".

Following Drinfeld [D1] , let us write a differential equation

$$\frac{dG(x)}{dx} = \hbar\left(\frac{A}{x} + \frac{B}{x-1}\right)G(x) \tag{3.6}$$

Let G_1 and G_2 be solutions of (3.6) defined when $0 < x < 1$ with the asymptotic behavior

$$G_1(x) \approx x^{\hbar A} (x \to 0)$$

and

$$G_2(x) \approx (x-1)^{\hbar B}(x \to 1)$$

Then

$$G_1 = G_2 \phi_{kz} \tag{3.7}$$

for some formal noncommutative power series ϕ_{kz} .

Theorem 3.2 (Drinfeld). *ϕ_{kz} is an "associator".*

Everywhere below we'll fix some choice of "associator" ϕ once and for all (for instance , let us put $\phi = \phi_{kz}$). All our constructions will work for any choice of ϕ .

We'll need for our purposes to define a semi-direct product Y^m of the group algebra KS_m of the symmetric group S_m , and A^m_{kz} as follows: Y^m is generated as a linear space by pairs (x,s) , where x is diagram from A^m_{kz} ; $s \in S_m$. Multiplication on Y^m is defined as follows:

$$(x_1, s_1)(x_2, s_2) = (s_2(x_1)x_2, s_1 s_2)$$

Here we suppose that the symmetric group acts on A^m_{kz} by permutations of strings. Algebra Y^m has an important subgroup

$$G^m = S_m * exp(X^m)_1 Y^m.$$

Let $s_i (1 \leq i \leq m-1)$ be the standard generators of the braid group B_m satisfying relations

$$s_i s_j = s_j s_i \qquad if(i-j) > 1, \tag{3.8}$$

and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{3.9}$$

(if it will not lead to confusion, we'll denote the elementary transpositions $s_i \in S_m$ by the same symbols as the braid group generators).

Let us define a representation ρ: $B_m \to G^m{}_1Y^m$ as follows:

$$\rho(s_1) = (e^{\frac{ht^{12}}{2}}; s_1), \tag{3.10}$$

$$\rho(s_i) = \phi^{-1}(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1})(e^{\frac{ht^{i,i+1}}{2}}; s_i)\phi(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}) \tag{3.11}$$

if $1 < i$.

This construction of representation ρ is due to Drinfeld (the second formula in the proof of proposition 5.1. of [Dr2]). It may be called "the universal prounipotent" braid group representation since the group G^m can be interpreted as a prounipitent completion of B_m .

We'll prove in this section why ρ really gives us a braid group representation since it was not explained in [Dr2] or anywhere else. Representation ρ

constructed above is a generalization of the braid group action on quasitensor category [Re1].

To construct representation of the braid group B_m one has to choose some configuration of parentheses in the (nonassociative) product of m symbols $x_1, ..., x_m$. Each transition from one configuration of parentheses to another configuration of parentheses can be decomposed (in a non-unique way) in the product of "the elementary transitions" of the form (3.12) where only one pair of parentheses changes:

$$...((x_i...x_{j-1})((x_j...x_{k-1})(x_k...x_{l-1})))... \to ...(((x_i...x_{j-1})(x_j...x_{k-1}))(x_k...x_{l-1}))... \tag{3.12}$$

Let us associate to "the elementary transition" (3.12) "the elementary transition operator" Φ_{ijkl}

$$\Phi_{ijkl} = \phi(h\sum_{s=i}^{j-1}\sum_{p=j}^{k-1} t^{s,p}; h\sum_{p=j}^{k-1}\sum_{r=k}^{l-1} t^{p,r}) \tag{3.13}$$

Then , to any transition from one configuration of parentheses to another configuration of parentheses we can associate "transition operator" Φ_{trans} by functoriality. The "pentagon identity" (3.2) insures that Φ_{trans} is independent of the choice of decomposition in the product of the elementary transitions.

Then, in order to define the action of the braid group generator s_i, we should:

a) change the configuration of parentheses in order to have $...(x_i x_{i+1})...$ inside one pair of parantheses (this gives us some "transition operator" Φ_{trans}),

b) apply the Drinfeld's R-matrix $(e^{\frac{ht^{i,i+1}}{2}}; s_i)$, and

c) return back to our initial configuration of parentheses (this gives us an inverse operator to the operator Φ_{trans}) .

Formulas (3.10) and (3.11) correspond to one particular choice of configuration of parentheses, namely, $(...((x_1x_2)...)x_{m-1})x_m)$ but any other choice is possible as well and gives us an equivalent representation with the transition operator between these two configurations of parentheses as an intertwinier. (If it will not lead to confusion, we'll denote all "transition operators" corresponding to transitions between different configurations of parentheses, by the same symbol Φ_{trans}).

Lemma 3.3. *If* $(i-j) > 1$ *then the equations* (3.14) – (3.17) *hold:*

$$[t^{i,i+1}; t^{j,j+1}] = 0 \tag{3.14}$$

$$[t^{i,i+1}; \phi(h\sum_{p=1}^{j-1} t^{p,j}; ht^{j,j+1})] = 0 \tag{3.15}$$

$$[t^{j,j+1}; \phi(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1})] = 0 \tag{3.16}$$

$$[\phi(h\sum_{s=1}^{i-1}t^{s,i};ht^{i,i+1});\phi(h\sum_{p=1}^{j-1}t^{p,j};ht^{j,j+1})]=0 \tag{3.17}$$

Proof: Relations (3.14) and (3.15) follow directly from (2.6).

Relation (3.16) follows from the fact that $[t^{j,j+1};h\sum_{s=1}^{i-1}t^{s,i}]=0$, from (3.14) and from the Leibnitz rule.

Relation (3.17) follows from (3.14),(3.15),(3.16), from the fact that

$$[\sum_{s=1}^{i-1}t^{s,i};\sum_{p=1}^{j-1}t^{p,j}]=0 \tag{3.18}$$

and from the Leibnitz rule.

To prove (3.18) it is sufficient to notice that $[\sum_{s=1}^{i-1}t^{s,i};t^{p,j}]=0$ for any p and then take the sum over the index p. The lemma is proved.

Lemma 3.4. *If* $(i-j)>1$ *then:*

$$\rho(s_i)\rho(s_j)=\rho(s_j)\rho(s_i) \tag{3.19}$$

Proof: It follows immediately from the definition of ρ given by (3.10) and (3.11) , and from lemma 3.3.

Lemma 3.5.

$$\rho(s_1)\rho(s_2)\rho(s_1)=\rho(s_2)\rho(s_1)\rho(s_2) \tag{3.20}$$

Proof: If we use the definition of ρ *given by (3.10) and (3.11) , then (3.20) can be rewritten in the following form:*

$$e^{\frac{ht^{12}}{2}}(\Phi^{213})^{-1}e^{\frac{ht^{13}}{2}}\Phi^{231}e^{\frac{ht^{23}}{2}}=(\Phi^{123})^{-1}e^{\frac{ht^{23}}{2}}\Phi^{132}e^{\frac{ht^{13}}{2}}(\Phi^{312})^{-1}e^{\frac{ht^{12}}{2}}\Phi^{321}. \tag{3.20A}$$

Using (3.5) several times and multiplying both. the l.h.s and the r.h.s of (3.20A) by Φ on the right we obtain another equivalent form of (3.20):

$$e^{\frac{ht^{12}}{2}}\Phi^{312}e^{\frac{ht^{13}}{2}}(\Phi^{132})^{-1}e^{\frac{ht^{23}}{2}}\Phi=\Phi^{321}e^{\frac{ht^{23}}{2}}(\Phi^{132})^{-1}e^{\frac{ht^{13}}{2}}\Phi^{213}e^{\frac{ht^{12}}{2}} \tag{3.20B}$$

Using (3.3) we see that the l.h.s of (3.20B) is equal to $(e^{\frac{ht^{12}}{2}};s_1)(e^{\frac{ht^{13}+ht^{23}}{2}};1)$ and the r.h.s of (3.20B) is equal to $(e^{\frac{ht^{13}+ht^{23}}{2}};1)(e^{\frac{ht^{12}}{2}};s_1)$.

The equality of these two expressions follows from (2.6B) which proves the lemma.

Let $\Phi_i=\phi(h\sum_{p=1}^{i-1}t^{p,i}+t^{p,i+1};ht^{i,i+2}+ht^{i+1,i+2})\phi(h\sum_{s=1}^{i-1}t^{s,i};ht^{i,i+1})$.

Lemma 3.6. *The equations* (3.21) *and* (3.22) *hold:*

$$\Phi_i\rho(s_i)\Phi_i^{-1}=(e^{\frac{ht^{i,i+1}}{2}};s_i) \tag{3.21}$$

$$\Phi_i \rho(s_{i+1})\Phi_i^{-1} = \phi^{-1}(ht^{i,i+1}; ht^{i+1,i+2})(e^{\frac{ht^{i+1,i+2}}{2}}; s_{i+1})\phi(ht^{i,i+1}; ht^{i+1,i+2}) \tag{3.22}$$

Proof: Since

$$[(e^{\frac{ht^{i,i+1}}{2}}; s_i); \phi(h\sum_{p=1}^{i-1} t^{p,i} + t^{p,i+1}; ht^{i,i+2} + ht^{i+1,i+2})] = 0$$

then $\Phi_i^{-1}(e^{\frac{ht^{i,i+1}}{2}}; s_i)\Phi_i$ is equal to the r.h.s of (3.11). Thus,

$$\rho(s_i) = \Phi_i^{-1}(e^{\frac{ht^{i,i+1}}{2}}; s_i)\Phi_i$$

which is equivalent to (3.21).

To prove (3.22) let us use "the pentagon identity" (3.2) in the form

$$\phi(ht^{i,i+1}; ht^{i+1,i+2})\phi(h\sum_{p=1}^{i-1} t^{p,i} + t^{p,i+1}; ht^{i,i+2} + ht^{i+1,i+2})\phi(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}) =$$

$$= \phi(h\sum_{r=1}^{i} t^{r,i+1} + t^{r,i+2}; ht^{i,i+1} + ht^{i,i+2})\phi(h\sum_{s=1}^{i-1} t^{s,i+1}; ht^{i+1,i+2}) \tag{3.23}$$

or equivalently

$$\phi(ht^{i,i+1}; ht^{i+1,i+2})\Phi_i =$$

$$= \phi(h\sum_{r=1}^{i} t^{r,i+1} + t^{r,i+2}; ht^{i,i+1} + ht^{i,i+2})\phi(h\sum_{s=1}^{i-1} t^{s,i+1}; ht^{i+1,i+2}) \tag{3.23A}$$

(3.23A) implies that

$$\Phi_i^{-1}\phi^{-1}(ht^{i,i+1}; ht^{i+1,i+2})(e^{\frac{ht^{i+1,i+2}}{2}}; s_{i+1})\phi(ht^{i,i+1}; ht^{i+1,i+2})\Phi_i =$$

$$= \phi^{-1}(h\sum_{s=1}^{i-1} t^{s,i+1}; ht^{i+1,i+2})\phi^{-1}(h\sum_{r=1}^{i} t^{r,i+1}+t^{r,i+2}; ht^{i,i+1}+ht^{i,i+2})(e^{\frac{ht^{i+1,i+2}}{2}}; s_{i+1})$$

$$\phi(h\sum_{r=1}^{i} t^{r,i+1} + t^{r,i+2}; ht^{i,i+1} + ht^{i,i+2})\phi(h\sum_{s=1}^{i-1} t^{s,i+1}; ht^{i+1,i+2}) \tag{3.24}$$

Since

$$[(e^{\frac{ht^{i+1,i+2}}{2}}; s_{i+1}); \phi(h\sum_{r=1}^{i} t^{p,i+1} + t^{p,i+2}; ht^{i,i+1} + ht^{i,i+2})] = 0$$

then the r.h.s of (3.24) can be rewritten in the form

$$\phi^{-1}(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2})(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1})\phi(h\sum_{s=1}^{i-1}t^{s,i+1};ht^{i+1,i+2}) \quad (3.24A)$$

But the expression (3.24A) is equal to $\rho(s_{i+1})$ which implies

$$\Phi_i^{-1}\phi^{-1}(ht^{i,i+1};ht^{i+1,i+2})(e^{\frac{ht^{i+1,i+2}}{2}};s_{i+1})\phi(ht^{i,i+1};ht^{i+1,i+2})\Phi_i =$$

$$= \rho(s_{i+1}) \quad (3.25)$$

But (3.25) is equivalent to (3.22). The lemma is proved.

Lemma 3.7.

$$\rho(s_1)\rho(s_{i+1})\rho(s_i) = \rho(s_{i+1})\rho(s_i)\rho(s_{i+1}) \quad (3.26)$$

Proof: (3.21) and (3.22) reduce the statement of the lemma to the case $i = 1$. But this case was already proved in the the lemma 3.5. The lemma is proved.

4. TAKING THE TRACE

It is well-known (see, for instance, [Bi]) that any oriented s-component link L can be presented as a closed braid. Two braids $b_1 \in B_{m1}$ and $b_2 \in B_{m2}$ give under closure the same link iff they can be obtained from each other by a finite sequence of Markov moves of two types:

$$b_1b_2 \approx b_2b_1 \in B_m \quad (4.1)$$

and

$$b \in B_m \approx bs_m^{\pm 1} \in B_{m+1} \quad (4.2)$$

Thus, any function $f : \bigcup_m B_m \rightarrow \bigcup_s F^s$ gives rise to some link invariant iff f takes equal values on braids equivalent with respect to (4.1) and (4.2).

Any framed link also can be presented as a closed braid. The analogues of Markov moves for braids which give under closure the same framed link (with blackboard framing [Tu], [Pi3], [Pi4]) can also be described explicitly (see [Re2]). Here we give sufficient conditions (4.1A) and (4.2A) for a function $f : \bigcup_m B_m \rightarrow \bigcup_s F^s$ to descend to some framed link invariant:

$$f(b_1b_2) = f(b_2b_1) \qquad (b_1; b_2 \in B_m), \quad (4.1A)$$

and

$$f(bs_m^{\pm 1}) = q^{\pm 1} * f(b) \qquad b \in B_m \quad (4.2A)$$

Here $*$ is the action of F on F^s (on the s-th component) , $q = e^{\frac{ht}{2}} \in F$, t is the standard generator in F_1 (see fig.8).

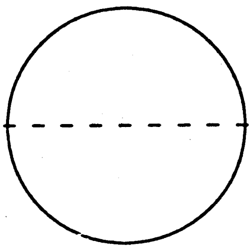


fig.8

Let us fix a configuration of parentheses in the (nonassociative) product of $2m$ symbols $x_1, ..., x_m, y_m, ..., y_1$ as follows :

$$((x_1((x_2(...((x_{m-1}(x_m y_m))y_{m-1})...))y_2))y_1) \tag{4.3}$$

Let us define, using this configuration of parantheses, formulas (3.10) – (3.13) and remarks following them, a representation $\widehat{\rho}$: $B_m \to G^{2m}$ as the restriction to $B_m \imath B_{2m}$ of the representation

$$\Phi^{-1}_{trans}\rho\Phi_{trans} : B_{2m} \to G^{2m}\imath Y^{2m},$$

where Φ_{trans} is the transition operator between "the standard" configuration of parentheses on the set of $2n$ elements and the configuration (4.3).

Now let us suppose that the first m Wilson lines in any "diagram" $(x, s) \in Y^{2m}$ are oriented "up" and the second m wilson lines are oriented "down".Then for any $m \imath N$ let us consider a map $\tau : Y^{2m} \to \bigcup_{s=1}^{m} F^s$ of graded linear spaces, defined as follows:

For any diagram $(x, s) \in Y^{2m}$ we have $4m$ free ends on it . Let us mark each of these free ends with a natural number from 1 to m as it is shown on fig.9a.

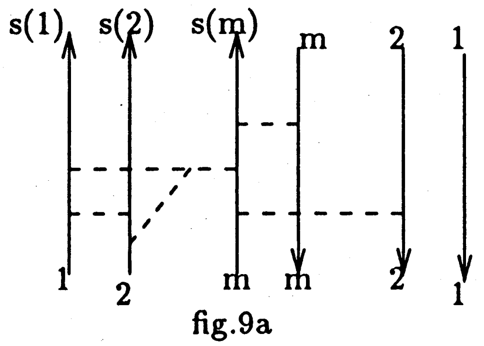


fig.9a

Then let us connect by (directed) line each pair of free ends on the top of the diagram with the same markings , and let us do the same on the bottom of the diagram (see fig.9b as an example. In this example $m = 3$)

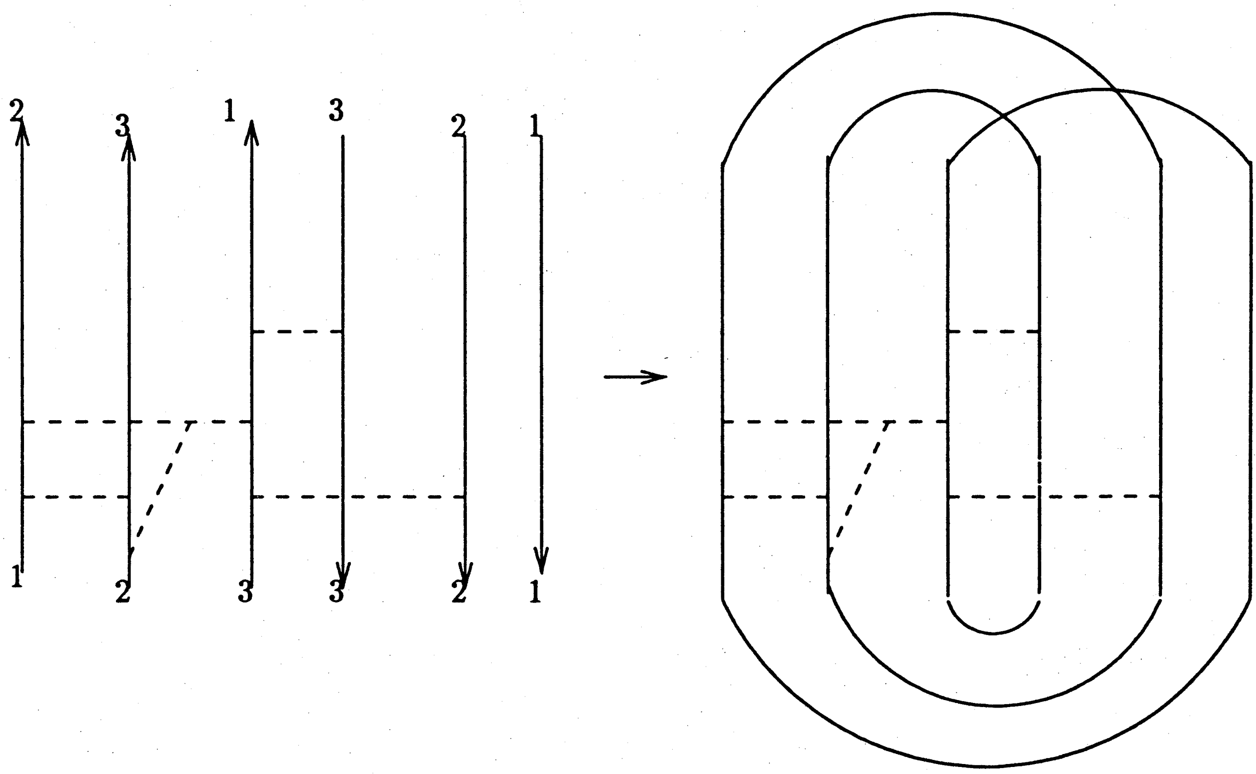


fig.9b

The result of this procedure will be, by definition, $\tau(x, s)$.

If it will not lead to confusion, we will not distinguish braids in B_m and their images in Y^{2m}.

Let $b_1 \in B_{m_1}$; $b_2 \in B_{m_2}$ be two braids, let b_2 gives a knot under closure, and let $(b_1 * b_2) \in B_{m_1+m_2-1}$ be the braid, obtained from b_1 and b_2 by the procedure shown on fig.10.

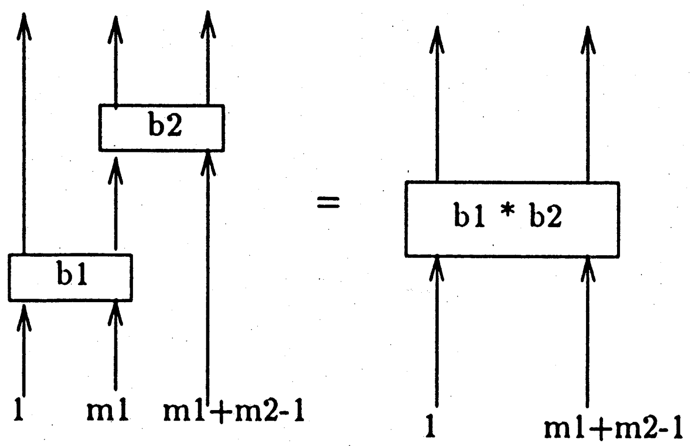


fig.10

Theorem 4.1.

$$\tau(b_1 * b_2) = \tau(b_2) * \tau(b_1) \tag{4.4}$$

where $*$ is the action of F on F^s (on the s-th component).

Proof. geometrically obvious from (4.3),fig.9b and fig.10.

Let $q = e^{\frac{ht}{2}}$ and let $\mu \in F$ be the image of associator $\Phi \in exp(X^3)\imath UX^3$ under "the closure map" shown on fig.11.

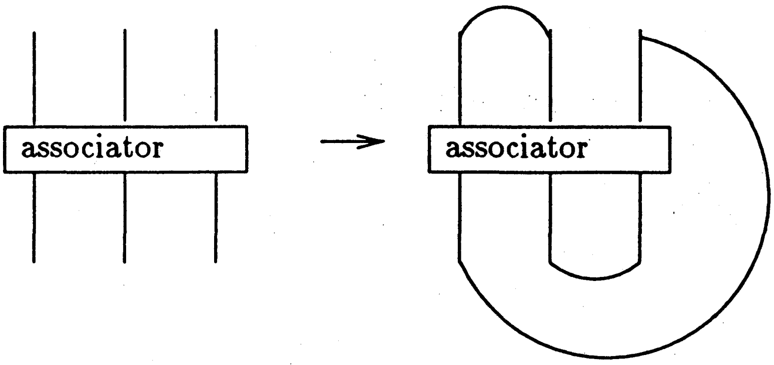


fig.11

Remark.

If $\Phi = \Phi_{kz}$, then μ is equal to the value of the generating function of Kontsevich integrals on the Morse knot shown on fig.12.

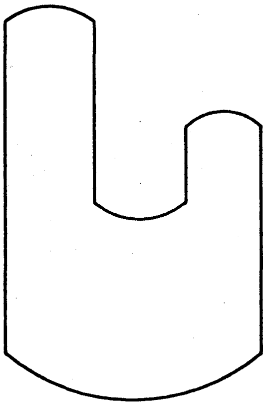
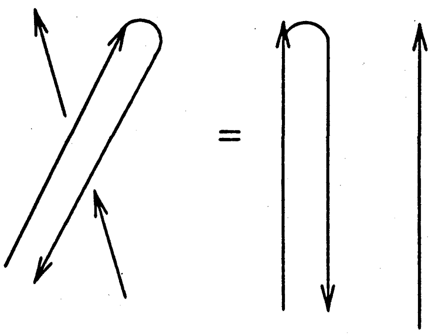


fig.12

Lemma 4.2. *The identity (4.5) shown on fig.13 holds:*



(4.5)

fig.13

Proof: It follows from (3.3) that the l.h.s of (4.5) is equal to the expression shown on fig.14

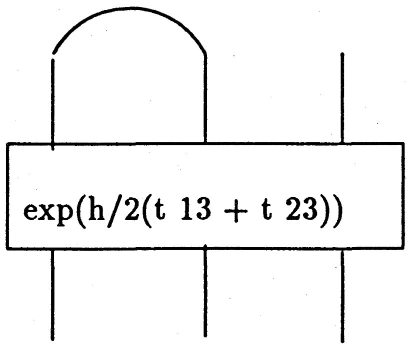
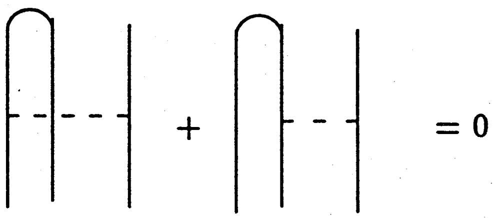


fig.14

But this expression vanishes since (4.6) holds



(4.6)

fig.15

Lemma 4.3. *Let s_1 be the standard generator of B_2. Then $\tau(s_1^{\pm 1}) = q^{\pm 1}\mu$*

We give here a pictorial proof:

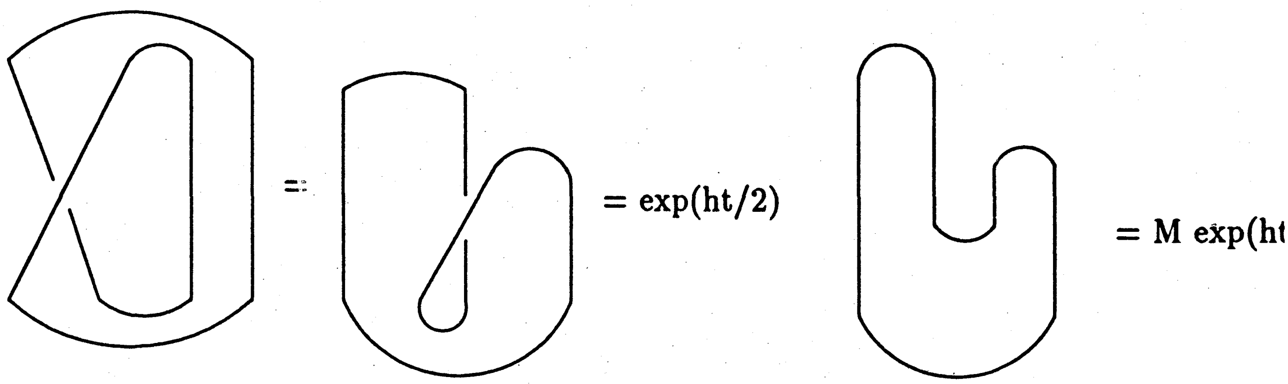


fig.16

The first identity in fig.16 follows from lemma 4.2.

Let $P : B_m \to \bigcup_s F^s$ be equal to $(\mu)^{1-m}\tau : B_m \to \bigcup_s F^s$.

Lemma 4.4. *The map P is a "Markov trace" i.e., it satisfies (4.1A) and (4.2A).*

Property (4.1A) is geometrically obvious. Property (4.2A) follows from the theorem 4.1 and from the lemma 4.3.

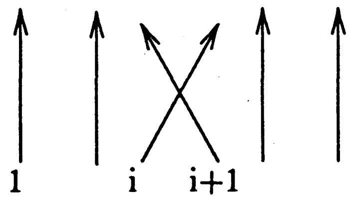
Let P be above defined framed link invariant. Let us consider its perturbative expansion:$P = \sum_{n=0}^{\infty} h^n P_n$

Lemma 4.5. *P_n is F_n^s - valued Vassiliev framed link invariant of order n.*

Proof: Let $b \in B_m$ be a braid and let $\widehat{\rho}(b) = \sum_{n=0}^{\infty} x_n(b) h^n \imath Y^{2m}$.

Then $x_n(b)\imath Y^{2m}$ has degree n in Y^{2m} (this fact is true for the generators $s_i \in B_m$ and thus, for any $b \in B_m$) .Thus , for any framed oriented link L , $P_n(L)$ also has degree n , which implies $P_n(L) \in F_n^s \imath F^s$.

Let L be a singular imbedding of $(S^1)^s$ into R^3 with $(n+1)$ double crossing points. Then L can be presented as a closure of a "generalized braid" [Pi1], [Ba] (braid where in some places the generators s_i are changed to the generators a_i with double crossings on $i-th$ place. The generators a_i are depicted on fig.17.)



(4.7)

fig.17

The representation $\rho : B_m \to Y^m$ can be extended to these "generalized braids" by the formula

$$\rho(a_i) = \rho(s_i) - \rho(s_i^{-1}) \tag{4.8}$$

(and the representation $\widehat{\rho} : B_m \to Y^{2m}$ can also be extended to "the generalized braids" by the same formula).

(4.8), (3.10) and (3.11) imply that

$$\rho(a_1) = (2sh\frac{ht^{12}}{2}; s_1), \tag{4.9}$$

and

$$\rho(s_i) = \phi_{kz}^{-1}(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1})(2sh\frac{ht^{i,i+1}}{2}; s_i)\phi_{kz}(h\sum_{s=1}^{i-1} t^{s,i}; ht^{i,i+1}) \tag{4.10}$$

if $1 < i$

Thus $\widehat{\rho}(a_i)$ are divisible by h in $Y^{2m} \otimes C[h]$. YThis fact implies, that for any "generalized braid" $b \in B_m$ with $(n+1)$ double crossing points , $\widehat{\rho}(b)$ is divisible by h^{n+1} . Thus $P(L)$ is divisible by h^{n+1} , which means that $P_n(L) = 0$ for any singular embedding L with $(n+1)$ double crossing points, or, equivalently, that P_n is a Vassiliev invariant of order n . The lemma is proved.

Let V_n^s be the space of Vassiliev invariants of framed s-component links of order n. Then there is a natural map $f_n : V_n^s \to V_n^s/V_{n-1}^s \to (F_n^s)^*$, defined as follows: Let v be some Vassiliev invariant of order n and let D be Vassiliev $[n]$-diagram. Then

$$(P_n(v); D) = (v; L(D)), \tag{4.11}$$

where $L(D)$ is some singular embedding $(S^1)^s \to R^3$ with n double crossing points for which the underlying configuration of n points on $(S^1)^s$ is given by the diagram D.

Theorem 4.6.

The map $< P_n; ... >: (F_n^s)^* \to V_n^s$ is left inverse to f_n, and differs from its right inverse on some Vassiliev invariant of order $n-1$.

Proof. It is sufficient to prove that for any singular embedding $L : (S^1)^s \to R^3$ with precisely n double points equation (4.12) holds:

$$P_n(L) = D(L), \tag{4.12}$$

where $D(L)$ a is $CS-$diagram with n propagators, joining those points on $(S^1)^s$, which are identified under L.

Let us present L as a closure of some "generalized braid" $b \in B_m$. Then $\widehat{\rho}(b)$ is product of some terms of the form

$$(e^{\frac{ht^{i,i+1}}{2}}; s_i), \tag{4.13}$$

$$\Phi_{trans}^{\pm 1}, \tag{4.14}$$

and

$$(2sh\frac{ht^{i,i+1}}{2}; s_i) \tag{4.15}$$

There are precisely n terms of forms of the form (4.15).

Since the following statements hold:

a) the terms (4.13) and (4.14) have the form

$$1 + hX \tag{4.16}$$

for some $X \in Y^{2m}$;

b) $\mu^{\pm 1}$ also has the form (4.16) for some $X \in F$; and

c) the terms (4.15) have the form

$$ht^{i,i+1} + h^2 X \tag{4.17}$$

for some $X \in Y^{2m}$,

then the expression for the coefficient in h^n in perturbative expansion of $\widehat{\rho}(b)$ consists of the single term. This term is the product of n terms of the form (4.15) . This fact implies that $P_n(L) = D(L)$, as desired. The theorem is proved.

Kontsevich Hopf algebra F has a (graded) quotient $A = F/F_1F$. Then A_n^* is canonically identified with the space of Vassiliev unframed knot invariants of order n factored by the space of invariants of order $n-1$. In the basis of Vassiliev $[n]$-diagrams in F_n the projector $Pr : F_n \longrightarrow A_n$ can be described explicitly [Pi2].

$$Pr(D) = \sum_{k=0}^{n}(-t)^n \sum_{I} D_I, \tag{4.18}$$

where t is the generator of F_1; the second sum in (4.18) is taken over all $[k]$-subdiagrams D_I of D. The quantity $Pr(P) = \sum_{n=0}^{\infty} h^n Pr(P_n)$ which is the map : $Knots \longrightarrow F$ is the generating function for "universal" (order n)-Vassiliev knot invariants and has the same formal properties as the generating function $I = \sum_{n=0}^{\infty} h^n I_n$ of Kontsevich integrals [Ko1].

Theorem 4.7. *Let K_1 and K_2 be two oriented framed knots; $K_1 * K_2$ be their connected sum. Then $P(K_1 * K_2) = P(K_1)P(K_2)$.*

Proof. It follows immediately from theotem 3.1 and the definition of P .

5. DISCUSSIONS

At the moment , there are three different expressions for the universal Vassiliev knot invariant, (the quantity, which satisfies conditions of theorems 4.3 and 4.6). The first one is constructed from perturbative expansion of monodromy of KZ-equation (Kontsevich integrals [Ko1]), the second one is constructed from perturbative Chern-Simons theory [Ko1], [BN3] (see also [AS] and [GMM]). The third construction is presented here (see also [Ko5] where a similiar combinatorial construction was given, using knot diagram and a point on it).

The "universal Vassiliev invariant" in the form presented here can be evaluated purely combinatorially for any particular link L , if we know an explicit expression for the "Drinfeld's associator" ϕ_{kz} as a formal noncommutative power series in $\hbar t^{12}$ and $\hbar t^{23}$ . . An "iterated integral" expression for the "assosiator" was proposed in [BN6], which proves immediately the equivalence of our approach with Kontsevich's one (see also [LM] for some related results).

The analogous problem for "Kontsevich integrals" is much more complicated and involves calculations with hypergeomertic type integrals [TK] and with polylogarithms [Ao].In our approach only 2^n such integrals (for each $n \in N$) should be calculated. The calculations in perturbative Chern-Simons theory are even more complicated and are hardly to be accomplished by direct methods.

Above defined construction τ of the universal Vassiliev invariant of a link which can be presented as a closure of braid has a straightforvard gneralization to an arbitrary link diagram, and even to a string link diagram [BN5]. Roughly speaking, τ is a decomposition of the generating function of Kontsevich integrals (before inserting the correction factor μ^{1-m}) in the product of "the elemrntary" factors corresponding to the decomposition of the link diagram into "the elementary" pieces.

Drinfeld's construction of the representation ρ: $B_m \rightarrow G^m$ depends on the choice of "associator" ϕ. We can construct explicitely only one such "associator" (namely, ϕ_{kz}) but we would like construct explicitely "the uiversal Q-valued Vassiliev invariant".

There are two possibilities how one could do this. The first one is to try to calculate explicitely Q-valued "associator" (which nobody knows how to do). The second possibility is to prove that all these formally different "universal Vassiliev invariants" (with different ϕ) are equal. We conjecture that it is so.

Acknowledgements

I benefited much from discussions with D.Bar-Natan, V.Drinfeld, D.Kazhdan and expecially with M.Kontsevich to whom I express my sincere gratitude.

Literature

[Ao] K.Aomoto. Special values of hyperlogarithms and linear difference schems. Illinois J. Math., v.34(1990), p.191-216.

[AS] S.Axelrod, I.Singer. Chern-Simons perturbation theory. MIT preprint, October 1991.

[Ar1] V.I.Arnold. Private communication.

[Ar2] V.I.Arnold. Lecture given on the European Congress of Mathematics. July 1992, Paris.

[Ba] J.C.Baez, Link invariants of finite type and perturbation theory. Lett.Math. Phys., v.26 (1992), p.43-51.

[Bi] J.S.Birman, Braids, links and mapping class groups. Ann.of Math.Studies, no 82, Princeton Univ.Press. Princeton, NJ, 1974.

[BL] J.S.Birman and X.-S.Lin, Knot polynomials and Vassiliev's invariants. Inv.Math. v.111 (1993) Fasc.2, p.225-270.

[BN1] D.Bar-Natan, On Vassiliev knot invariants. to appear in Topology.

[BN2] D.Bar-Natan, Weights of Feynmann diagrams and Vassiliev knot invariants. Preprint.February 1991.

[BN3] D.Bar-Natan, Perturbative aspects of the Chern-Simons topological quantum field theory.Ph.D.thesis, Princeton Univ., June 1991, Dep.of Mathematics.

[BN4] D.Bar-Natan, Perturbative Chern-Simons theory. Preprint. August 1990.

[BN5] D.Bar-Natan, Vassiliev homotopy string link invariants. Preprint.December 1992.

[BN6] D.Bar-Natan, Private communication.

[Co] J.H.Conway, An enumeration of Knots and Links and some of their Algebraic Properties. in Computational Problems Abstract Algebra, p.329-358, New York, Pergamon Press, 1969.

[Dr1] V.G.Drinfeld, Quasi-Hopf algebras,
Leningrad Math J.,vol.1 (1990)p.1419-1457.

[Dr2] On quasitriangular quasi-Hopf algebras and a group, closely connected with Gal(Q/Q). Leningrad Math J.,vol.2 (1991)p.829-860.

[Dr3] V.G.Drinfeld, Quantum groups, Proc. Intern. Congr.Math. Berkeley, 1(1986), p.798-820.

[GMM] E.Guadagnini, M.Marellini and M.Mintchev,
Perturbative aspects of the Chern-Simons field theory. Phys.Lett. B227(1989)p.11

[HOMFLY]P.Freyd, D.Yetter, J.Hoste, W.B.R.Lickorish, K.Milett, A.Ocneanu. A new polynomial invariant of knots and links.
Bull. Amer. Math., Soc.(1985), v.12, No2, p.239-246.

[Jo1] V.F.R.Jones, A polynomial invariant for links via von Neumann algebras. Bull. AMS, v.12(1985), p.103-110.

[Jo2] V.F.R.Jones, On knot invariants related to some statistical mechanical models. Pacific J.Math. v.137 (1989)p.311-334.

[K1] T.Kohno. Monodromy representations of braid groups and classical Yang-Baxter c equations, Ann.Inst.Fourier, Grenoble, v.37(1987), p.139-160.

[K2] T.Kohno. Series de Poincare-Koszul associe aux grouppes de tresses pures. Inv.Math. No 82(1985), p.57-75

[Ka1] L.H.Kauffman, An invariant of regular isotopy. Trans.Amer.Math.Soc., vol.318, 2 (1990) 317-371.

[Ka2] L.H.Kauffman, State model and the Jones polynomial. Topology, v.26(1987)p.395-407.

[Ko1] M.Kontsevich, Vassiliev's knot invariants. Preprint 1992 .

[Ko2] M.Kontsevich, Private communication.

[Ko3] M.Kontsevich, Feynman diagrams and Low-dimensional topology.
Preprint, January 1993.

[Ko4] M.Kontsevich, Graphs, homotopical algebra and low-dimensional topology.
Pre-preprint, 1992.

[Ko4] M.Kontsevich, in preparation.

[KZ] V.G.Knizhnik and A.B.Zamolodchikov, Current algebra and Wess-Zumino models in two dimensions, Nucl.Phys.B247 (1984)p.83-103.

[Li1] X.S.Lin, Vertex models, quantum group and Vassiliev knot invariants. Preprint, 1991

[Li2] X.S.Lin, Milnor μ-invarianys ar all of finite type. Preprint, 1992.

[Mi1] J.Milnor, Link groups. Ann.of Math.59(1954), p.177-195.

[LM] Le Tu Quoc Thang and J.Murakami. On relation between Kontsevich's integral invariant and invariants coming from quantum R-matrices.
Pre-preprint.January 1993.

[Mi2] J.Milnor, Isotopy of links, Algebraic geometry and Topology (R.H.Fox ed.) Princeton Univ.Press., Princeton, No1, (1957) p.280-306.

[Pi1] S.A.Piunikhin, Weights of Feynmann diagrams, link polynomials and Vassiliev knot invariants. To appear in Journal of Knot Theory and its Ramifications .
[Pi2] S.A.Piunikhin, Vassiliev knot invariants contain more information, than all knot polynomials (To appear in Journal of Knot Theory and its Ramifications)
[Pi3] S.A.Piunikhin, Turaev-Viro and Kauffman-Lins Invariants for 3-Manifolds coincide. Journal of Knot Theory and its Ramifications,
vol 1, No2 (1992), p.105-135.
[Pi4] S.A.Piunikhin, State sum model for trivalent knotted graph invariants using the quantum group SL (2).
Journal of Knot Theory and its Ramifications , v.1, No3(1992), p.273-278.
[Ro] D.Rolfsen, Knot and Links, (Publish or Perish, 1976).
[Re1] N.Yu.Reshetikhin, Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links. LOMI-preprint E-4-87, E-17-87.
[Re2] N.Yu.Reshetikhin, Quasitriangular Hopf algebras and invariants of links. Leningrad, Math J.1 (1990)p.
[RT] N.Yu.Reshetikhin and V.G.Turaev, Ribbon graphs and their invariants derived from quantum groups. Commun.Math.Phys., vol.127, 1(1990) 1-26.
[TK] A.Tsuchiya, Y.Kanie Vertex operators in two dimensional conformal field theory on P^1 and monodromy representations of braid groups.
in Advanced studies in Pure Mathematics. 16. Tokyo, Kinokuniya, 1988, p.297-372.
[Tu] V.G.Turaev, The Yang-Baxter equation and invariants of links.
Invent.Math.v.92(1988)p.527-553.
[Tu] V.G.Turaev, Operator invariants of tangles and R-matrices, Math.UssR, Izvestiya, vol.35., (1990), No2, p.411-444.
[Va1] V.A.Vassiliev, Cohomology of Knot Spaces. Advances in Soviet Mathematics, v.1, AMS(1990), p.23-69 .
[Va2] V.A.Vassiliev, Complements of discriminants of smooth maps. Topology and applications, Trans.of Math.Mono.98, Amer.Math. Soc., Providence, 1992.
[Wi] E.Witten, Quantum field theory and Jones polynomial.
Commun.Math.Phys., vol.121, No3 (1989) p.351-399.